Diagonalization

As mentioned earlier, an $n \times n$ matrix D is a <u>diagonal matrix</u> if all the entries ~~off~~ the diagonal are 0.
That is,

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & \lambda_n \end{bmatrix} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \mathrm{diag}(1, 2, 5)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \mathrm{diag}(0, 2)$$

Calculations w/ diagonal matrices are easy:
If $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $E = (\alpha_1, \ldots, \alpha_n)$, then

1. $D + E = \mathrm{diag}(\lambda_1 + \alpha_1, \ldots, \lambda_n + \alpha_n)$
2. $DE = \mathrm{diag}(\lambda_1 \alpha_1, \ldots, \lambda_n \alpha_n)$
3. $\det D = \lambda_1 \lambda_2 \cdots \lambda_n$.

Def: An $n \times n$ matrix A is <u>diagonalizable</u> if

$P^{-1}AP$ is diagonal for some $n \times n$ invertible matrix P. P is called a <u>diagonalizing matrix</u> for A.

We can describe when A is diagonalizable using eigenvectors:

Theorem: Let A be an $n \times n$ matrix.

1.) A is diagonalizable if and only if it has eigenvectors $\vec{x}_1, \dots, \vec{x}_n$ such that the matrix

$$P = [\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n] \text{ is invertible.}$$

columns

2.) In this case, $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, where for each i, λ_i is the eigenvalue corresponding to $\vec{x}_i$.

Ex: (3.3.1c)

$$A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$$

$$c_A(x) = \det(xI - A) = \det \begin{bmatrix} x-7 & 0 & 4 \\ 0 & x-5 & 0 \\ -5 & 0 & x+2 \end{bmatrix}$$

$$= (x-5)\big((x-7)(x+2) + 20\big) \quad \text{(expand along row 2)}$$

$$= (x-5)(x^2 - 5x - 14 + 20)$$

$$= (x-5)(x^2-5x+6)$$
$$= (x-5)(x-3)(x-2)$$

Eigenvalues : $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 2$

$\lambda_1 = 5$:
$$\left[\begin{array}{ccc|c} -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & 0 & 7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Eigenvectors : $t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$

$\lambda_2 = 3$:
$$\left[\begin{array}{ccc|c} -4 & 0 & 4 & 0 \\ 0 & -2 & 0 & 0 \\ -5 & 0 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Eigenvectors : $t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$

$\lambda_3 = 2$:
$$\left[\begin{array}{ccc|c} -5 & 0 & 4 & 0 \\ 0 & -3 & 0 & 0 \\ -5 & 0 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -4/5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Eigenvectors : $t\begin{bmatrix} 4/5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$

or, equivalently : $t\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}, \quad t \neq 0.$

Take $P = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$. $\det P = -1(5-4) = -1$

so P is invertible.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 0 & 1 & 0 \\ 5 & 0 & -4 \\ -1 & 0 & 1 \end{matrix}}_{P^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & -4 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 \\ 15 & 0 & -12 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$$

In each of the examples we've seen so far, each eigenvalue has had only <u>one</u> basic eigenvector. Here's an example where that's not the case:

Ex: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$c_A(x) = \det(xI - A) = \det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix}$$

$$= x(x^2-1) + 1(-x-1) - 1(1+x)$$
$$= x(x+1)(x-1) - (x+1) - (x+1)$$
$$= (x+1)(x(x-1) - 1 - 1)$$
$$= (x+1)(x^2 - x - 2)$$
$$= (x+1)(x-2)(x+1)$$
$$= (x+1)^2(x-2)$$

so $\lambda_1 = -1$, $\lambda_2 = 2$ are the two eigenvalues.

$\lambda_1 = -1$:
$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

solution:

$$x = -s - t$$
$$y = s$$
$$z = t$$

$$\rightsquigarrow \quad s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

two basic eigenvectors

$\lambda_2 = 2$:
$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

solutions: $t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and $P^{-1}AP = \operatorname{diag}(-1,-1,2)$.

Def: An eigenvalue of A has <u>multiplicity</u> m if it occurs m times as a root of $c_A(x)$.

In the above example, $\lambda_1 = -1$ has multiplicity 2, and $\lambda_2 = 2$ has multiplicity 1. λ_1 had 2 basic eigenvectors, and λ_2 had 1.

Theorem: A is diagonalizable if and only if every eigenvalue λ of multiplicity m has exactly m basic eigenvectors; i.e. the solution of $(\lambda I - A)\vec{x} = \vec{0}$ has exactly m parameters.

Theorem: An $n \times n$ matrix w/ n distinct eigenvalues is diagonalizable.

How to diagonalize a matrix:

A an $n \times n$ matrix.

1. Find eigenvalues of A (roots of $c_A(x) = \det(xI - A)$)

2. Find basic eigenvectors for each eigenvalue (basic solutions of $(\lambda I - A)\vec{x} = \vec{0}$).

3. A is diagonalizable if and only if there are a total of n basic eigenvalues.

4. If A is diagonalizable, then $P^{-1}AP$ is diagonal.
(P = matrix w/ (multiples of) basic eigenvectors as columns, and $P^{-1}AP$ has diagonal corr. eigenvalues.)

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $\quad c_A(x) = \det \begin{bmatrix} x-1 & -1 \\ 0 & x-1 \end{bmatrix} = (x-1)^2$

eigenvalue: $\lambda = 1$, mult $= 2$.

$$\left[\begin{array}{cc|c} 1-1 & -1 & 0 \\ 0 & 1-1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

solution: $t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. only 1 basic eigenvector.

So A is not diagonalizable.

Ex: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\quad c_A(x) = \det \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} = x^2+1$.

No real roots, so no real eigenvalues, so A is not diagonalizable. (except with complex #s)

Bird pop. example:

We can now finish the example from the beginning of the section:

$\vec{v}_k = \begin{bmatrix} a_k \\ j_k \end{bmatrix}$, $\vec{v}_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$, and

$\vec{v}_{k+1} = A\vec{v}_k$, where $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$

We want to compute $\vec{v}_k = A^k \vec{v}_0$.

First, we diagonalize A:

$$C_A(x) = \det(xI - A) = \det\begin{bmatrix} x - 1/2 & -1/4 \\ -2 & x \end{bmatrix} = x(x - 1/2) - 1/2$$

$$= x^2 - \frac{1}{2}x - 1/2$$

$$= (x-1)(x + 1/2)$$

$\Rightarrow$ eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1/2$ (2 distinct eigenvalues $\Rightarrow$ diagonalizable)

$\lambda_1 = 1$: $\left[\begin{array}{cc|c} 1/2 & -1/4 & 0 \\ -2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

eigenvectors: $t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, or $t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for convenience.

$\lambda_1 = -1/2$: $\left[\begin{array}{cc|c} -1 & -1/4 & 0 \\ -2 & -1/2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

eigenvectors: $t\begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$, or $t\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

$P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$, so $P^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$

So $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} = D$

$\Rightarrow \quad A = PDP^{-1}$

So we can finally do our computation:

$$\vec{V}_k = A^k \vec{V}_0 = (PDP^{-1})^k \vec{V}_0$$

$$= (PD^kP^{-1})\vec{V}_0$$

$$= \left(\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (-1/2)^k \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \right) \vec{V}_0$$

$$= \frac{1}{6} \left(\begin{bmatrix} 1 & -(-1/2)^k \\ 2 & 4(-1/2)^k \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \right) \vec{V}_0$$

$$= \frac{1}{6} \begin{bmatrix} 4 + 2(-1/2)^k & 1 - (-1/2)^k \\ 8 - 8(-1/2)^k & 2 + 4(-1/2)^k \end{bmatrix} \begin{bmatrix} 100 \\ 40 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 400 + 200(-1/2)^k + 40 - 40(-1/2)^k \\ 800 - 800(-1/2)^k + 80 + 160(-1/2)^k \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 440 + 160(-1/2)^k \\ 880 - 640(-1/2)^k \end{bmatrix}$$

So $a_k = \frac{220}{3} + \frac{80}{3}\left(-\frac{1}{2}\right)^k$ and

$j_k = \frac{440}{3} - \frac{320}{3}\left(-\frac{1}{2}\right)^k$

So, for example, after 5 years,

$a_5 \approx 72.5$ and $j_5 = 150$

In the long run, as k gets large, $\left(-\frac{1}{2}\right)^k \longrightarrow 0$, so the adult pop. stabilizes at $\frac{220}{3} \approx 73.3$ and the juvenile pop. at $\frac{440}{3} \approx 146.7$.

Practice Problems: 3.3: 1adfgi, 6, 8a, 9, 13